Zoom Class 12/5/2020

Integration as a limit of a Sum.

- If the summation is of the form then:

1) $\lim\limits_{h\to 0} \sum\limits_{r=0}^{n-1} h f(a+rh) = \int_a^b f(x)\,dx$

where $nh = b-a$

2) $\lim\limits_{n\to\infty} \sum\limits_{r=0}^{n-1} \left[\frac{b-a}{n} f\left(a + r\cdot\frac{(b-a)}{n}\right)\right] = \int_a^b f(x)\,dx$

3) $\lim\limits_{n\to\infty} \sum\limits_{r=0}^{n-1} \left[\frac{1}{n} f\left(\frac{r}{n}\right)\right] = \int_0^1 f(x)\,dx$

4) $\lim\limits_{n\to\infty} \sum\limits_{r=1}^{n} \left[\frac{1}{n} f\left(\frac{r}{n}\right)\right] = \int_0^1 f(x)\,dx$

5) $\lim\limits_{n\to\infty} \sum\limits_{r=1}^{n} \left[\frac{b-a}{n} f\left(a + r\cdot\frac{(b-a)}{n}\right)\right] = \int_a^b f(x)\,dx.$

Working Rule

First we write the r^{th} term of the given series then transform the r^{th} term or t_r into any of the given above forms and obtain the respective integral. This integral is further solved and the value obtained is value of the series.

Ques Evaluate $\lim_{n\to\infty}\left\{\frac{n}{n^2+1^2}+\frac{n}{n^2+2^2}+\dots+\frac{n}{n^2+n^2}\right\}$

Soln. Here the r^{th} term is

$$t_r = \frac{n}{n^2+r^2} = \frac{1}{n}\left(\frac{n^2}{n^2+r^2}\right)$$

$$= \frac{1}{n}\left(\frac{1}{1+(r/n)^2}\right)$$

i.e., the series is of the form

$$\frac{1}{n}f\left(\frac{r}{n}\right)$$

and since the value of 'r' ranges from $r=1$ to $r=n$

we can write the series as

$$\lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{n}f\left(\frac{r}{n}\right) = \int_0^1 f(x)\,dx$$

$$= \int_0^1 \frac{1}{1+x^2}\,dx \quad \left\{\text{replacing } \left(\frac{r}{n}\right) \text{ with } x\right\}$$

$$= \left[\tan^{-1}(x)\right]_0^1 = \tan^{-1}1 - \tan^{-1}0$$

$$= \frac{\pi}{4} \text{ Ans}$$

Ques Evaluate $\lim\limits_{n\to\infty} \dfrac{1+\sqrt{2}+\sqrt{3}+\dots+\sqrt{n}}{n^{3/2}}$

Soln Given series is $\lim\limits_{n\to\infty} \dfrac{1+\sqrt{2}+\sqrt{3}+\dots+\sqrt{n}}{n^{3/2}}$

$$= \lim_{n\to\infty}\left\{\frac{\sqrt{1}}{n^{3/2}}+\frac{\sqrt{2}}{n^{3/2}}+\frac{\sqrt{3}}{n^{3/2}}+\dots+\frac{\sqrt{n}}{n^{3/2}}\right\} = \lim_{n\to\infty}\sum_{r=1}^{n}\frac{\sqrt{r}}{n^{3/2}}$$

$$\therefore\ t_r = \frac{\sqrt{r}}{n^{3/2}} = \frac{1}{n}\sqrt{\frac{r}{n}} = \frac{1}{n}f\left(\frac{r}{n}\right)$$

$$\therefore \lim_{n\to\infty}\sum_{r=1}^{n}\frac{\sqrt{r}}{n^{3/2}} = \lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{n}f\left(\frac{r}{n}\right)$$

$$= \int_0^1 f(x)\,dx$$

$$= \int_0^1 \sqrt{x}\,dx = \frac{2}{3}\left[x^{3/2}\right]_0^1 = \frac{2}{3}[1-0] = \frac{2}{3}$$ **Ans**

Ques Evaluate $\lim\limits_{n\to\infty}\sum\limits_{r=1}^{n}\dfrac{r^3}{r^4+n^4}$.

Soln Given series is $\lim\limits_{n\to\infty}\sum\limits_{r=1}^{n}\dfrac{r^3}{r^4+n^4} = \lim\limits_{n\to\infty}\sum\limits_{r=1}^{n}\dfrac{nr^3}{n(r^4+n^4)}$

$$= \lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{n}\frac{(r/n)^3}{(1+(r/n)^4)} = \lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{n}f\left(\frac{r}{n}\right)$$

$$= \int_0^1 f(x)\,dx = \int_0^1 \frac{x^3}{1+x^4}\,dx = \frac{1}{4}\int_0^1\frac{4x^3}{1+x^4}\,dx$$

$$= \frac{1}{4}\left[\log(1+x^4)\right]_0^1 = \frac{1}{4}[\log 2 - \log 1] = \frac{1}{4}\log\left(\frac{2}{1}\right)$$

$$= \frac{1}{4}\log 2$$ **Ans**

Ques Evaluate $\lim_{n\to\infty}\left\{\frac{n}{n^2}+\frac{n}{n^2+1^2}+\frac{n}{n^2+2^2}+\dots+\frac{n}{n^2+(n-1)^2}\right\}$

Soln Here $t_r = \frac{n}{n^2+r^2} = \frac{1}{n}\cdot\frac{n^2}{n^2+r^2} = \frac{1}{n}\left\{\frac{1}{1+(r/n)^2}\right\}$

Given limits of 'r' are $r=0$ to $r=n-1$

$\therefore$ Given series is $\lim_{n\to\infty}\sum_{r=0}^{n-1}\frac{1}{n}\left(\frac{1}{1+(r/n)^2}\right)$.

$$= \lim_{n\to\infty}\sum_{r=0}^{1}\frac{1}{n}f\left(\frac{r}{n}\right)$$

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$$= \int_0^1 f(x)\,dx$$

$$= \int_0^1 \frac{1}{1+x^2}\,dx$$

$$= \left[\tan^{-1}(x)\right]_0^1 = \tan^{-1}1 - \tan^{-1}0$$

$$= \frac{\pi}{4} \quad \text{Ans}$$